Curves and Surfaces

CS425: Computer Graphics I

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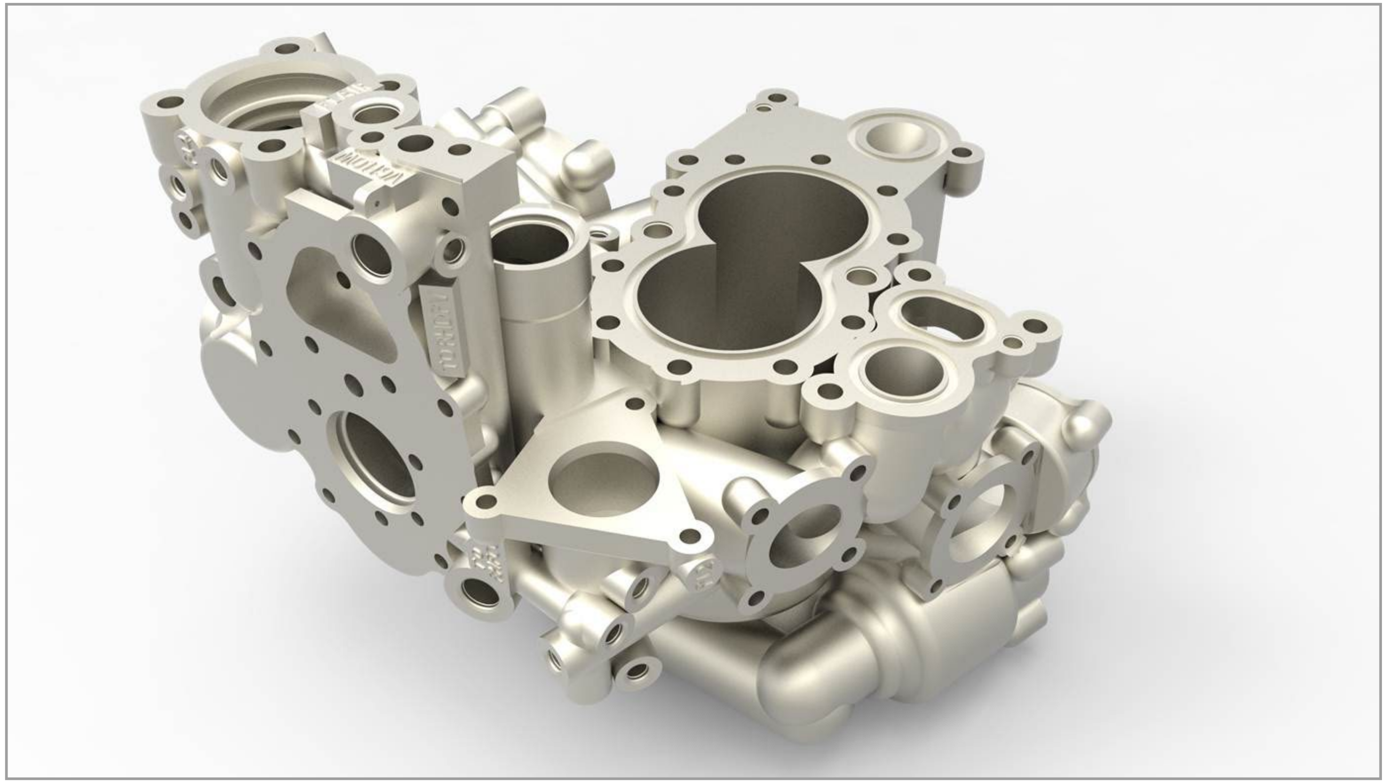
Overview

- Types of curves and surfaces:
  - Explicit
  - Implicit
  - Parametric
  - Bézier curves
- Modeling and approximations

How to model shapes?



From: Delcam Plc.

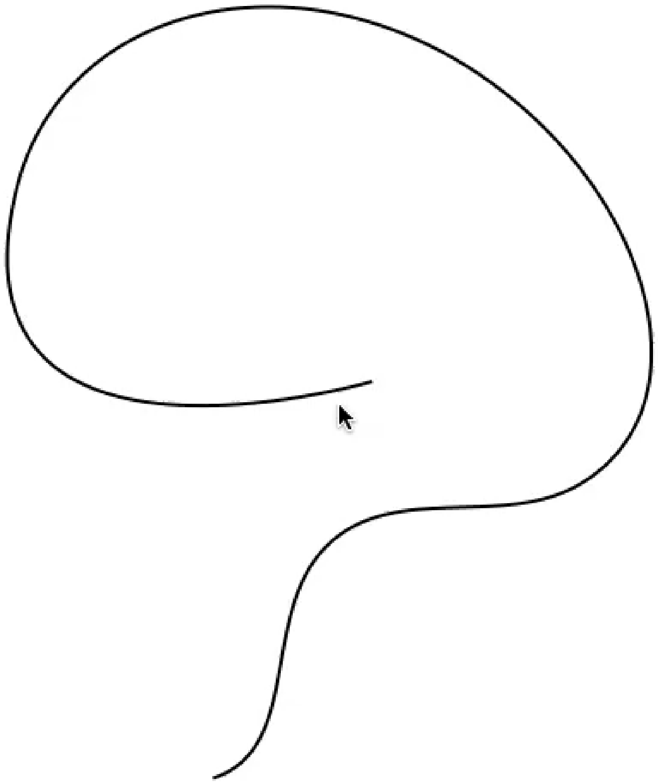
Beyond flatland

- Until now: flat entitles such as lines and polygons.
  - Flat entities fit well with graphics hardware and the graphics pipeline: texture mapping, hidden surface removal, etc.
  - Mathematically simple.
- World is not composed of flat entities:
  - Need curves and curved surfaces.
  - May only need them at the application level.
  - We can still render curves and curved surfaces approximating them with flat primitives.

Modeling curves

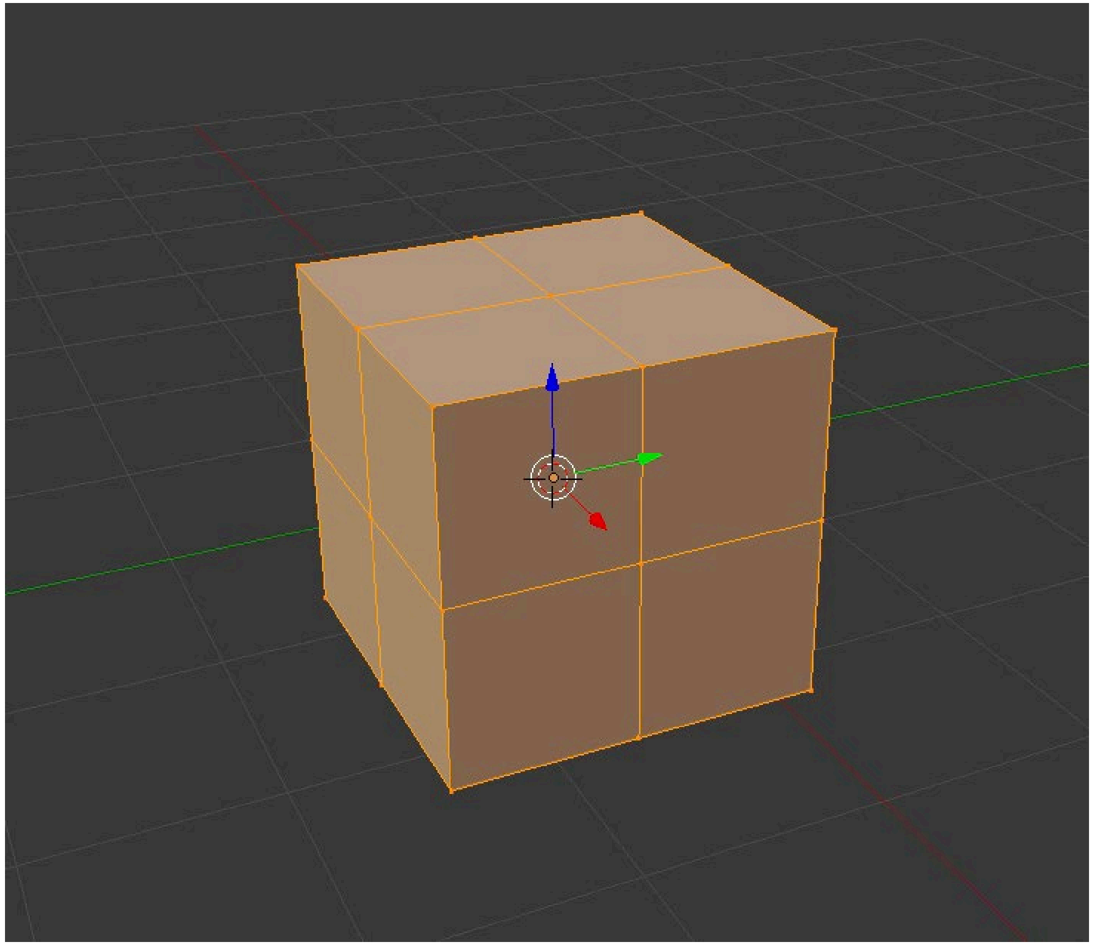
- We need mathematical concepts to characterize the desired curve properties.
- Curve geometry can help with designing user interfaces for curve creation and editing.
- Curves and surfaces are objects like meshes, but are expressed in terms of mathematical functions (rather than a series of discrete primitives).
  - Less memory at modelling time.
  - More work at rendering time.

Modeling in 2D



From: Daniele Panozzo - NYU

Modeling in 3D



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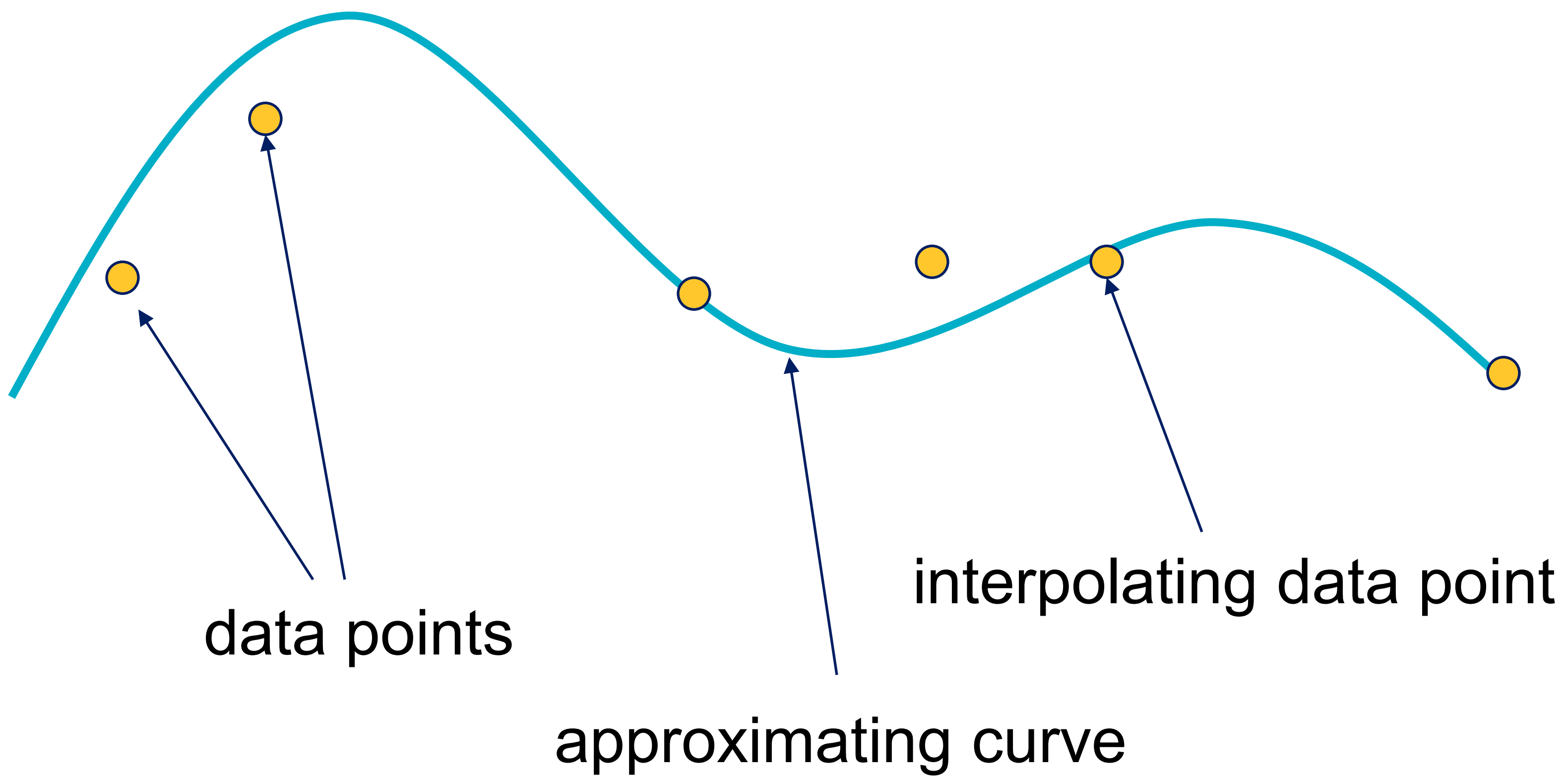
Modeling curves



Good representations

- Different ways to represent curves and surfaces.
- Representation goal:
  - Stable
  - Smooth
  - Easy to evaluate
  - Interpolate?
  - Derivatives?

Explicit representation

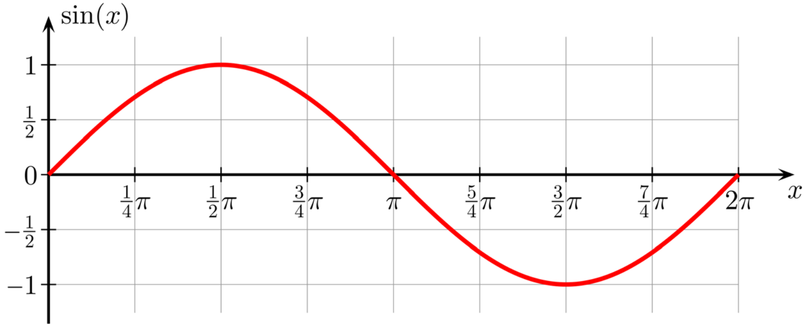
- Most familiar form of curve in 2D:

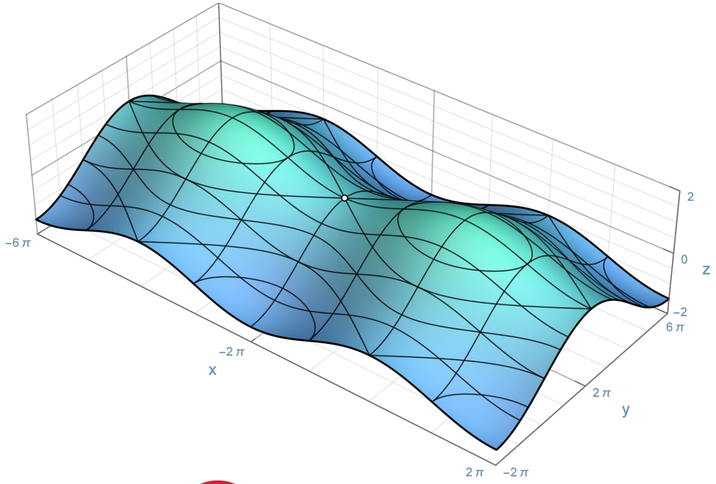
$$y = f(x)$$

- Cannot represent all curves:
  - Vertical lines
  - Circles
- Extension to 3D:

$$y = f(x), z = g(x)$$

$z = f(x, y)$ (defines a surface)





Implicit representation

- Two dimensional curves:

$$f(x, y) = 0$$

- Three dimensional surfaces:

$$f(x, y, z) = 0$$

- An implicit curve or surface is the set of zeros of a function of 2 or 3 variables.
- *Implicit*: equation is not solved for x or y or z.

Implicit representation

- Plane:

$$x + y - 3z + 1 = 0$$

- Sphere:

$$x^2 + y^2 + z^2 - 1 = 0$$

- Torus:

$$(x^2 + y^2 + z^2 + R^2 - a^2)^2 - 4R^2(x^2 + y^2) = 0$$

Implicit representation

- Function f is essentially a membership function that divides space into points that belong to the curve or surface and those that do not.
  - Take x, y pair and evaluate f to determine whether this point lies on the curve.
- No analytical way to find a value y on the curve that corresponds to a given x (or vice versa).
- Represents all lines and circles.

Implicit representation

$$f(x, y, z) = 0$$

$$x^2 + y^2 + z^2 - 9 = 0$$

$$(0,0,0) \rightarrow not\ on\ surface$$

$$(3,0,0) \rightarrow on\ surface$$

$$(0,3,0) \rightarrow on\ surface$$

$$(0,0,3) \rightarrow on\ surface$$

Algebraic surface

- Surface defined by an implicit equation $f(x, y, z) = 0$ where f is a polynomial in three indeterminates, with real coefficients.

$$\sum_i \sum_j \sum_k x^i y^j z^k = 0$$

- Quadric surfaces: each term can have degree up to 2 (spheres, disks, cones).

Parametric curves

- Separate equation for each spatial variable:

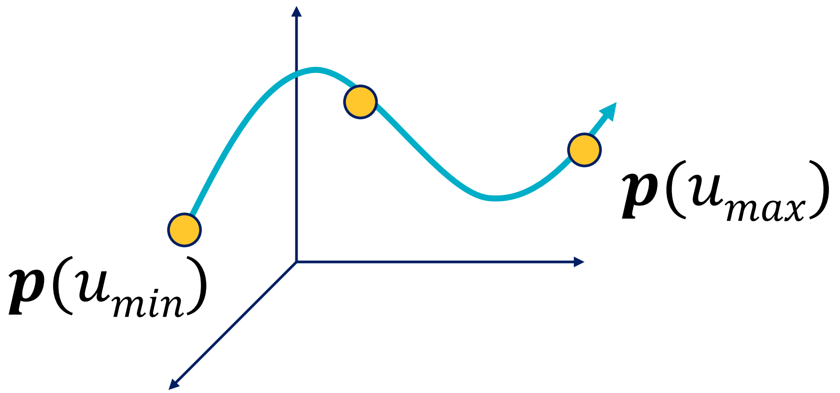
$$x = x(u)$$
$$y = y(u)$$
$$z = z(u)$$

- Each spatial variable on the curve is written in terms of an independent variable, or parameter, u.
- Useful representation (same in two and three dimensions).

$$p(u) = [x(u), y(u), z(u)]^T$$

Parametric curves

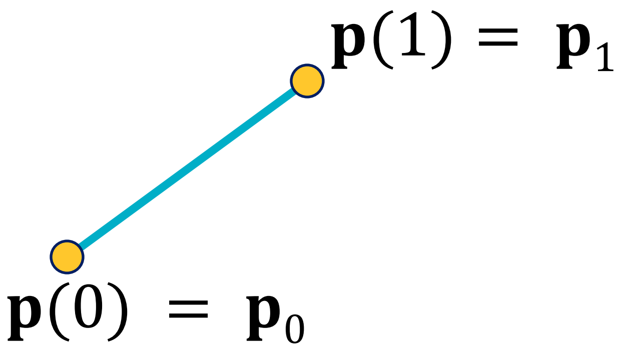
- For $u_{min} \leq u \leq u_{max}$, we trace out a curve in two or thee dimensions:



Parametric lines

- Line connecting two points p_0 and p_1:

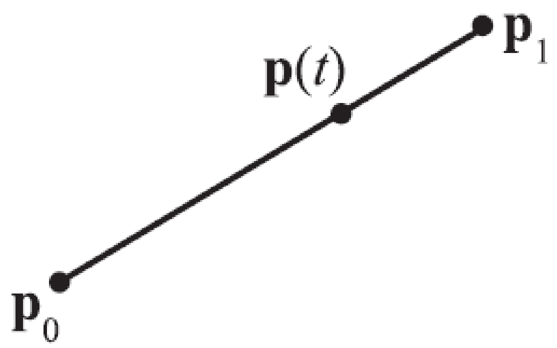
$$p(u) = p_0 + u(p_1 - p_0) = (1 - u)p_0 + up_1$$

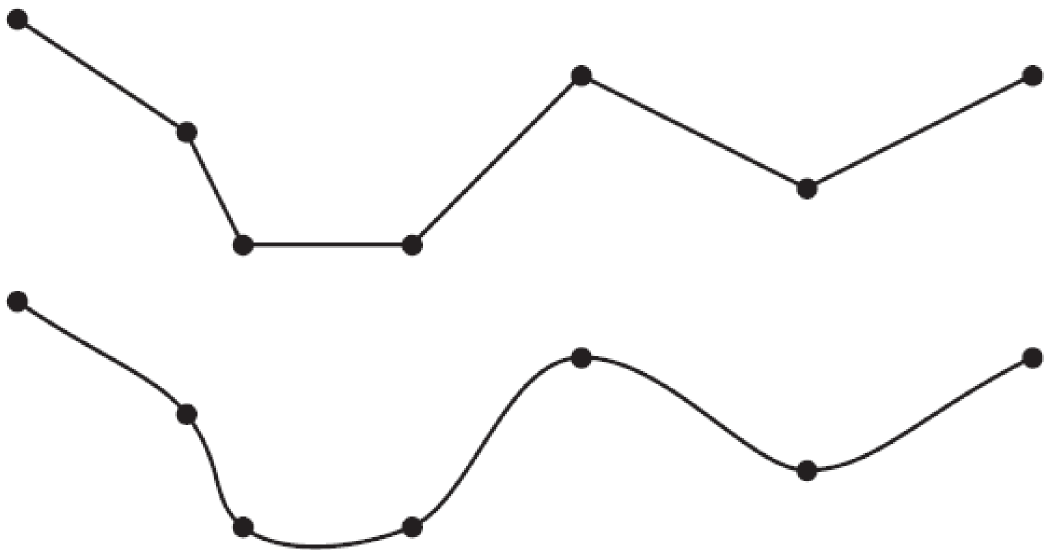


- Parameter u simply controls where on the line the point $p(u)$ will land.

Parametric lines

- When interpolating between only two points, linear interpolation might be enough. However, what if we have more points on a path?
  - Sudden changes at the points (joints) become unacceptable.



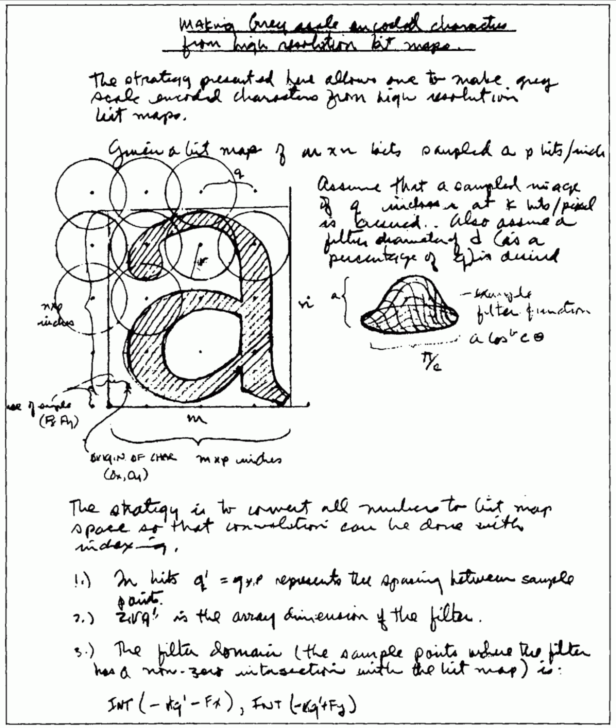


Bézier curves

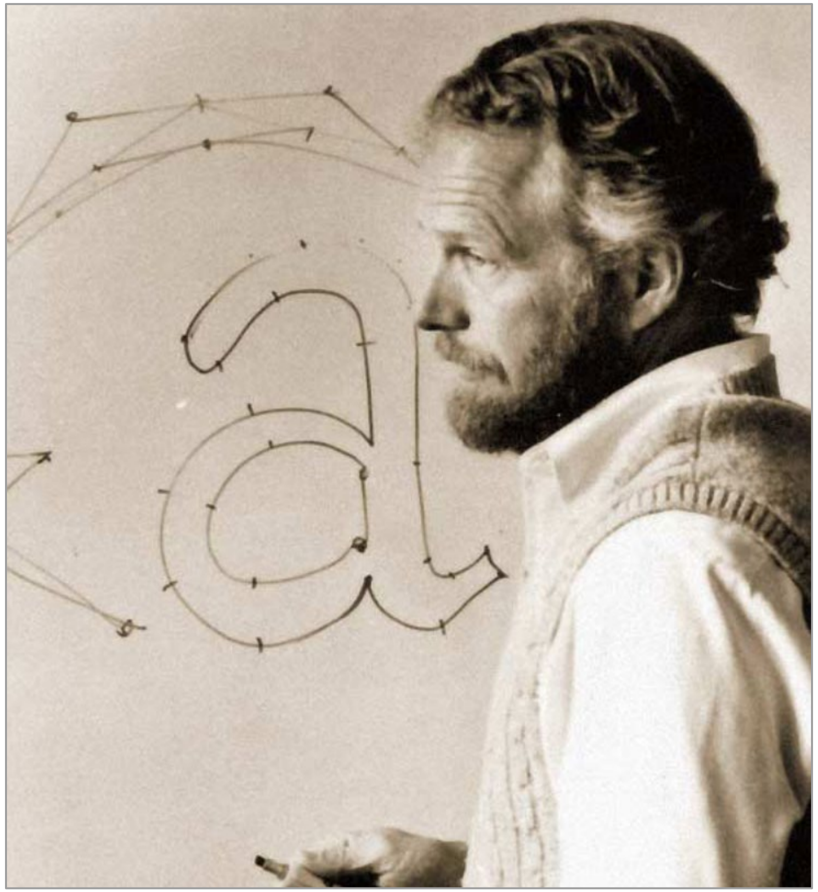
- Common form of parametric curves.
- Addresses discontinuous changes by applying repeated linear interpolations.
- Applications:
  - Animation: character movement
  - Games: camera movement
  - Graphics: model smooth curves
  - Fonts: PostScript, TrueType

Bézier curves

- PostScript, instead of requiring bitmaps to be generated for each style and size of typeface, generates fonts of any size and shape from Bézier curves.



Notebook page showing how to make gray-scale characters from high-resolution bit maps.



John Warnock

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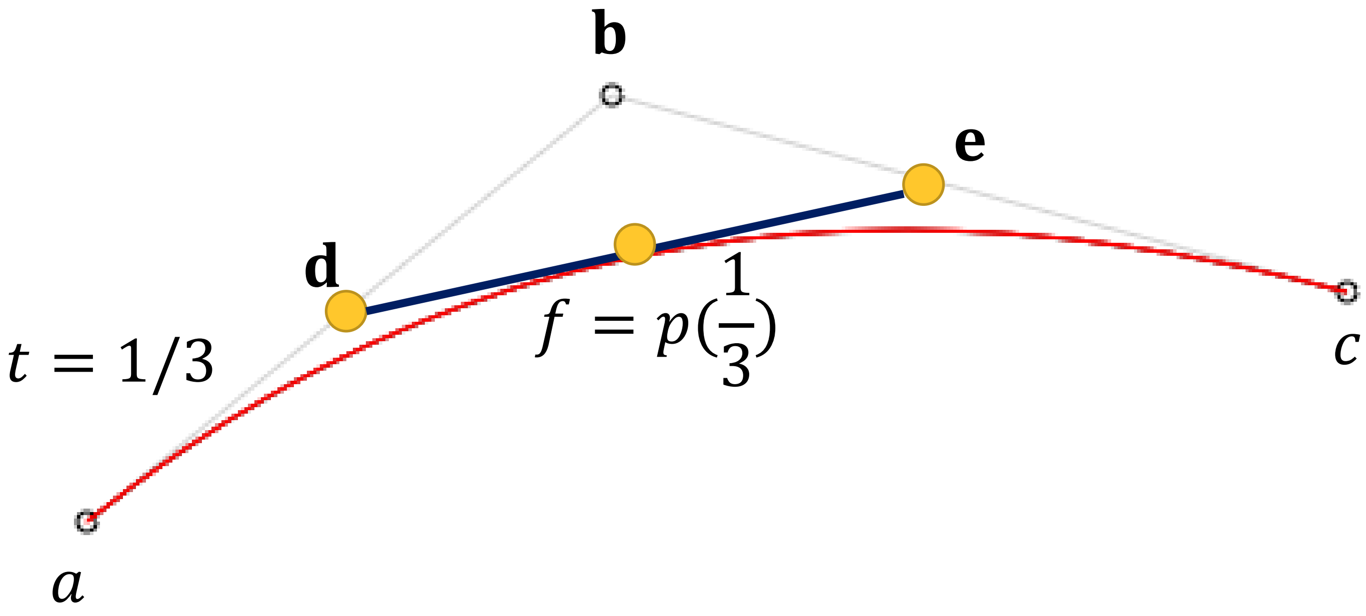
Bézier curves



- Three control points: a, b, c
- What is the point on the curve for the parameter $t = 1/3$?
  - Linearly interpolate between a and b to get d.
  - Linearly interpolate between b and c to get e.
  - Linearly interpolate between d and e to get final point $p\left(\frac{1}{3}\right) = f$.

Bézier curves

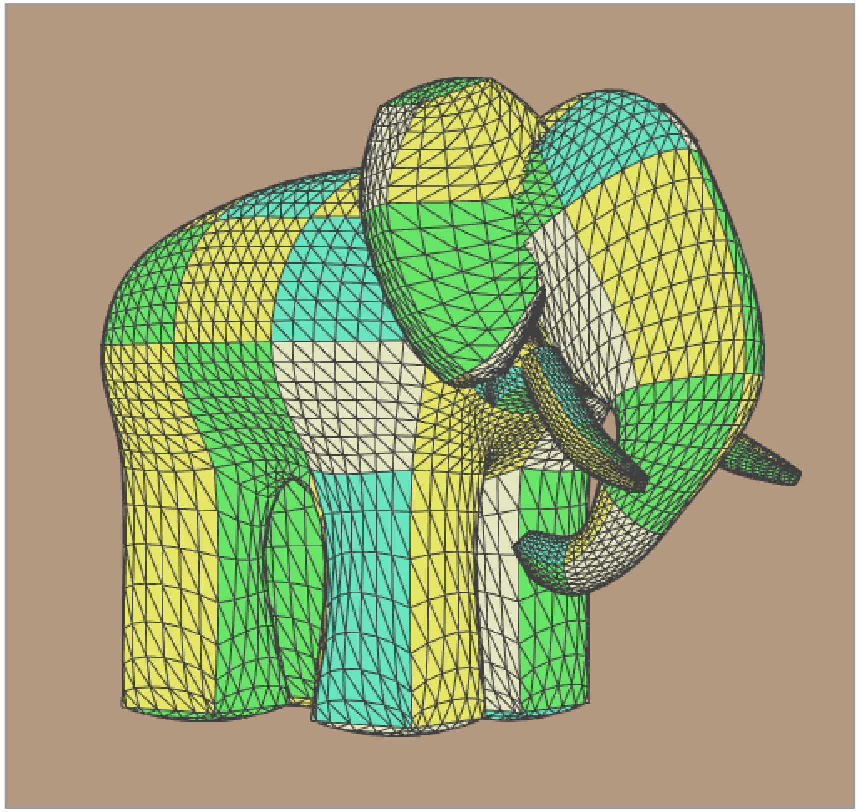
- Relationship:

$$p(t) = (1-t)\mathbf{d} + t\mathbf{e}$$
$$p(t) = (1-t)[(1-t)\mathbf{a} + t\mathbf{b}] + t[(1-t)\mathbf{b} + t\mathbf{c}]$$
$$p(t) = (1-t)^2\mathbf{a} + 2(1-t)t\mathbf{b} + t^2\mathbf{c}$$

- Parabola since the maximum degree of t is two.
- Given $n+1$ control points, the degree of the curve is n.

Bézier patches

- The same approach can be used in 3D: surface defined by a set of points in 3D.
- Superior to triangle meshes as a representation of smooth surfaces.



Ed Catmull's Gumbo model, composed from patches

Bézier patches

- Instead of using one parameter t, we now use two parameters (u, v).
- Using u to linearly interpolate between $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{c}$ and $\mathbf{d}$:

$$\mathbf{e} = (1-u)\mathbf{a} + u\mathbf{b}$$
$$\mathbf{f} = (1-u)\mathbf{c} + u\mathbf{d}$$

- Linearly interpolated points e and f are then interpolated in the other direction, using v.

$$p(u, v) = (1-v)\mathbf{e} + v\mathbf{f}$$
$$p(u, v) = (1-u)(1-v)\mathbf{a} + u(1-v)\mathbf{b} + (1-u)v\mathbf{c} + uv\mathbf{d}$$